The limits of cooperation. The Gaussian broadcast channel and Gaussian multiple access channel

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Outline

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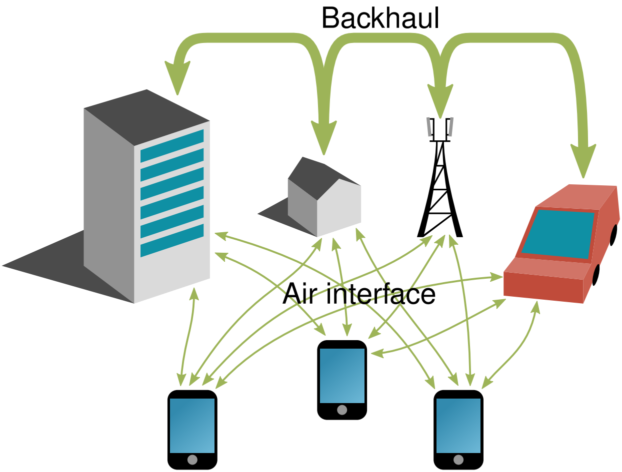
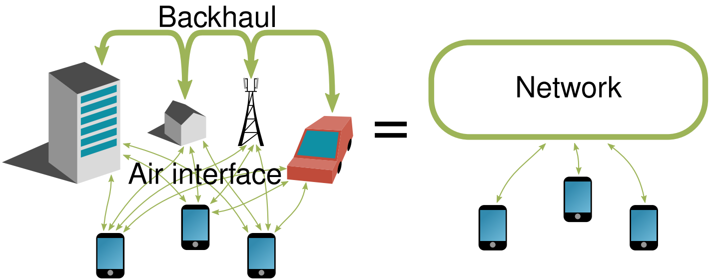
Introduction

- Which are the limits of cooperation?

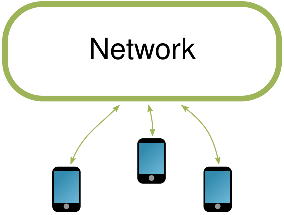


Introduction

- Utmost cooperation can be achieved with perfect backhaul (infinite capacity), processing power, and channel gains knowledge



Broadcast and multiple access channels



- The downlink direction is a broadcast channel
  - One transmitter sends information to several receivers
  - I will assume this information is independent
- The uplink direction is a multiple access channel
  - Several transmitters send independent information to a common receiver

Channel capacity

$$x \xrightarrow{f_h(y|x)} y$$

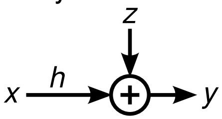
- The channel is a probability transition function $f_h(y|x)$
- The capacity of the channel is the maximum mutual information of the input and the output

$$C = \max_{f_x} I(x; y)$$

  - The maximization is done over all feasible input distributions f_x
  - An input distribution is feasible if it satisfies some (power) constraints

Gaussian channels

- The input is altered by an i.i.d. Gaussian noise



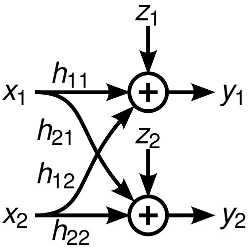
$$y = hx + z, \qquad z \sim \mathcal{N}(0, N), \; y \sim \mathcal{N}(hx, N)$$

- With an average power constraint of P, the capacity is (Shannon)

$$C = \max_{f_x | \mathrm{E}[xx^\dagger] \leq P} I(x; y) = \frac{1}{2} \log \left(1 + \frac{hh^\dagger P}{N} \right)$$

- This capacity is achieved when $x \sim \mathcal{N}(0, P)$, that is, when f_x is a Gaussian function with variance P

MIMO Gaussian channels



$\boldsymbol{y} = \boldsymbol{Hx} + \boldsymbol{z}$, all z_i are normally distributed

- Covariance matrix of the input signal: $\boldsymbol{Q} = \mathrm{E}[\boldsymbol{xx}^\dagger] \succeq 0$
- Noise covariance matrix: $\boldsymbol{Z} = \mathrm{E}[\boldsymbol{zz}^\dagger] \succ 0$
- Covariance matrix of the output signal: $\mathrm{E}[\boldsymbol{yy}^\dagger] = \boldsymbol{Z} + \boldsymbol{HQH}^\dagger \succ 0$

MIMO Gaussian channels

- Power constraints are given in terms of $\boldsymbol{Q}$
  - The average power of each antenna is in the diagonal of $\boldsymbol{Q}$

$$\boldsymbol{Q} = \mathrm{E}[\boldsymbol{x}\boldsymbol{x}^\dagger] = \begin{pmatrix} \mathrm{E}[x_1 x_1^\dagger] & \mathrm{E}[x_1 x_2^\dagger] \\ \mathrm{E}[x_2 x_1^\dagger] & \mathrm{E}[x_2 x_2^\dagger] \end{pmatrix}$$

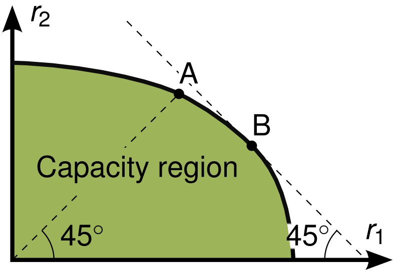
  - Let $\mathcal{P}$ be the set of feasible $\boldsymbol{Q}$
  - For a sum power constraint: $\mathcal{P} = \{\boldsymbol{Q} | \mathrm{tr}(\boldsymbol{Q}) \leq P, \boldsymbol{Q} \succeq 0\}$
- The capacity is

$$C = \max_{f_{\boldsymbol{x}} | \boldsymbol{Q} \in \mathcal{P}} I(\boldsymbol{x}; \boldsymbol{y}) = \frac{1}{2} \max_{\boldsymbol{Q} \in \mathcal{P}} \log \frac{|\boldsymbol{Z} + \boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^\dagger|}{|\boldsymbol{Z}|}$$

  - This capacity is achieved when $\boldsymbol{x}$ is normally distributed
  - For a sum power constraint, the maximum is achieved with waterfilling (Telatar [1])

Capacity region

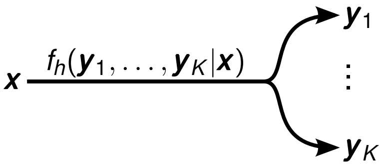
- With several users, we talk about a capacity region
- The rate vector $\boldsymbol{r}$ is achievable if user 1 can reach rate $r_1, \ldots,$ and user K can reach r_K.
- The capacity region is the closure of the set of all achievable rates



- Interesting points are in the boundary
  - A: Maximum $r_1 = r_2$
  - B: Maximum $r_1 + r_2$

Outline

Broadcast channel



- A broadcast channel is a probability transition function $f_h(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_K | \boldsymbol{x})$
- Expressions of the capacity region are not known
  - The largest known achievable region is due to Marton [2]

$$
\begin{aligned}
r_1 &\leq I(\boldsymbol{u}_1; \boldsymbol{y}_1) \\
r_2 &\leq I(\boldsymbol{u}_2; \boldsymbol{y}_2) \\
r_1 + r_2 &\leq I(\boldsymbol{u}_1; \boldsymbol{y}_1) + I(\boldsymbol{u}_2; \boldsymbol{y}_2) - I(\boldsymbol{u}_1; \boldsymbol{u}_2)
\end{aligned}
$$

  - for certain $f_{\boldsymbol{x}}(\boldsymbol{x}|\boldsymbol{u}_1, \boldsymbol{u}_2) f_{\boldsymbol{u}}(\boldsymbol{u}_1, \boldsymbol{u}_2)$ such that the marginal $f_{\boldsymbol{x}}(\boldsymbol{x})$ satisfies the power constraints
  - $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are auxiliary random variables

Degraded broadcast channel

$$\boldsymbol{x} \xrightarrow{f_{h_1}(\boldsymbol{y}_1|\boldsymbol{x})} \boldsymbol{y}_1 \xrightarrow{f_{h_2}(\boldsymbol{y}_2|\boldsymbol{y}_1)} \boldsymbol{y}_2 \cdots \boldsymbol{y}_{K-1} \xrightarrow{f_{h_K}(\boldsymbol{y}_K|\boldsymbol{y}_{K-1})} \boldsymbol{y}_K$$

- A broadcast channel is degraded if it can be written as

$$f_h(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_K|\boldsymbol{x}) = f_{h_1}(\boldsymbol{y}_1|\boldsymbol{x}) \prod_{k=2}^{K} f_{h_k}(\boldsymbol{y}_k|\boldsymbol{y}_{k-1})$$

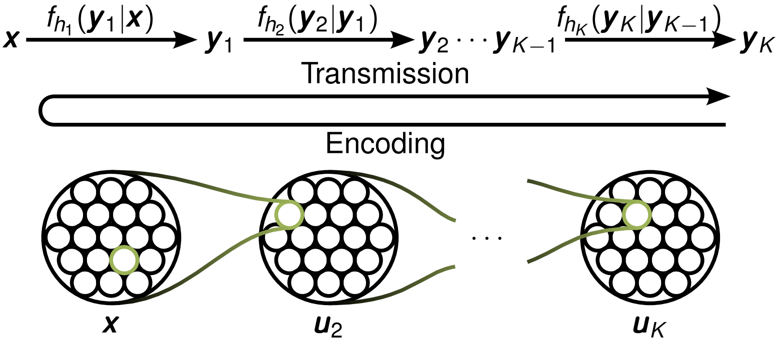
- The capacity region of these channels is

$$\begin{aligned} r_1 &\leq I(\boldsymbol{x}; \boldsymbol{y}_1|\boldsymbol{u}_2, \ldots \boldsymbol{u}_K) \\ r_k &\leq I(\boldsymbol{u}_k; \boldsymbol{y}_k|\boldsymbol{u}_{k+1}, \ldots \boldsymbol{u}_K), \quad k = 2, \ldots, K \end{aligned}$$

  - $\boldsymbol{u}_2, \ldots, \boldsymbol{u}_K$ are auxiliary random variables
  - This capacity is achieved with superposition coding

Degraded broadcast channel

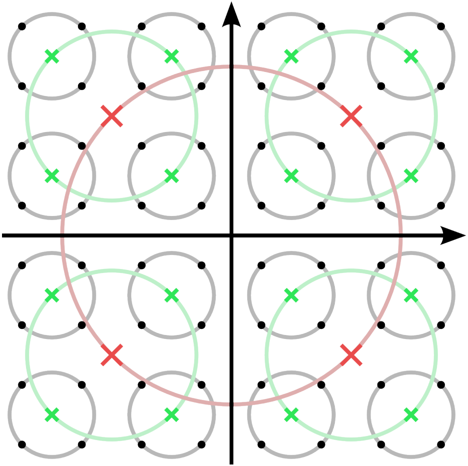
Superposition coding



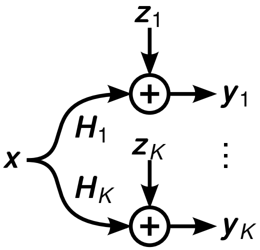
- Receiver k can see its own signal and the signal of all the receivers behind it

Degraded broadcast channel
Superposition coding

- Example of QAM and 3 users



Gaussian broadcast channel



- Number of users: K
- Signal devoted to user k: $\boldsymbol{x}_k$

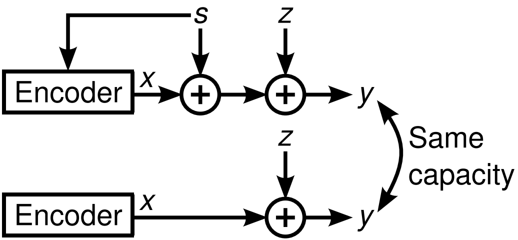
$$\boldsymbol{x} = \sum_{k=1}^{K} \boldsymbol{x}_k$$

$$\boldsymbol{y}_k = \boldsymbol{H}_k \boldsymbol{x} + \boldsymbol{z}_k$$

- $\boldsymbol{Q} = \mathrm{E}[\boldsymbol{x}\boldsymbol{x}^\dagger]$ must satisfy certain power constraints
- The capacity region was found in 2006 [3]
  - *Dirty paper coding* (DPC) and *time-sharing* achieve the capacity region

Dirty paper coding

- DPC uses Costa's result [4]
  - A Gaussian noise noncausally known at the encoder does not affect capacity



Dirty paper coding
How does DPC work?

1. Sort out users
2. Select a codeword for the first user
3. Select a codebook for the second user, taking into account the previous codeword
4. Select a codeword for the second user
5. Select a codebook for the third user, taking into account the two previous codewords
6. Select a codeword for the third user
7. ...

Dirty paper coding
How does DPC work?

- The first user is interfered by user 2,3,4,...
- The second user is interfered by user 3,4,...
- The third user is interfered by user 4,...

- User ordering is an important parameter of DPC
  - It has to be optimized
  - The optimum might be a combination of different orderings, which are shared in time (time-sharing)

Dirty paper coding

Achievable rates of one ordering

- K: Number of users
- $K!$: Number of user orderings
- $\boldsymbol{Q}_k = \mathrm{E}[\boldsymbol{x}_k \boldsymbol{x}_k^\dagger]$: Covariance matrix of the signal of user k
- $\boldsymbol{H}_k$: channel matrix of user k
- $\boldsymbol{Z}_k = \mathrm{E}[\boldsymbol{z}_k \boldsymbol{z}_k^\dagger]$: Noise covariance matrix at user k
- π: users permutation function, that is, $\pi(j)$ is the user in the j-th position
- Achievable rate of user k in ordering π:

$$r_k \leq \frac{1}{2} \log \frac{\left|\boldsymbol{Z}_k + \boldsymbol{H}_k \left(\sum_{j \geq \pi^{-1}(k)} \boldsymbol{Q}_{\pi(j)}\right) \boldsymbol{H}_k^\dagger\right|}{\left|\boldsymbol{Z}_k + \boldsymbol{H}_k \left(\sum_{j > \pi^{-1}(k)} \boldsymbol{Q}_{\pi(j)}\right) \boldsymbol{H}_k^\dagger\right|}$$

- Remember that $\boldsymbol{Q} = \sum_{k=1}^{K} \boldsymbol{Q}_k$ must satisfy certain power constraints

Dirty paper coding

Achievable rates of one ordering

- For the special case of the identity permutation function π^*, that is, $\pi^*(1) = 1, \ldots, \pi^*(K) = K$

$$r_1 \leq \frac{1}{2} \log \frac{\left| \boldsymbol{Z}_1 + \boldsymbol{H}_1 \left(\sum_{j=1}^{K} \boldsymbol{Q}_j \right) \boldsymbol{H}_1^\dagger \right|}{\left| \boldsymbol{Z}_1 + \boldsymbol{H}_1 \left(\sum_{j=2}^{K} \boldsymbol{Q}_j \right) \boldsymbol{H}_1^\dagger \right|}$$

$$r_2 \leq \frac{1}{2} \log \frac{\left| \boldsymbol{Z}_2 + \boldsymbol{H}_2 \left(\sum_{j=2}^{K} \boldsymbol{Q}_j \right) \boldsymbol{H}_2^\dagger \right|}{\left| \boldsymbol{Z}_2 + \boldsymbol{H}_2 \left(\sum_{j=3}^{K} \boldsymbol{Q}_j \right) \boldsymbol{H}_2^\dagger \right|}$$

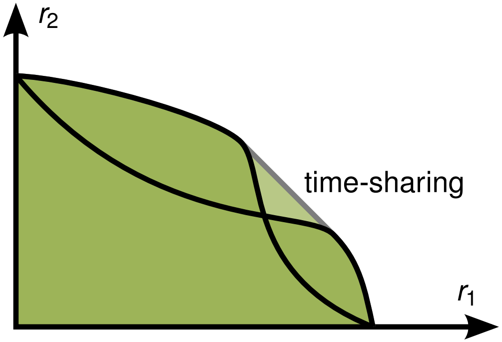
$$\vdots$$

$$r_K \leq \frac{1}{2} \log \frac{\left| \boldsymbol{Z}_K + \boldsymbol{H}_K \boldsymbol{Q}_K \boldsymbol{H}_K^\dagger \right|}{\left| \boldsymbol{Z}_K \right|}$$

Go to SIC $\rightarrow$

Dirty paper coding
Achievable rates of DPC

- DPC needs time-sharing to achieve the capacity region of the GBC
  - We just need a convex combination of $K+1$ rate vectors (Carathéodory's theorem)



Dirty paper coding
Achievable rates of DPC

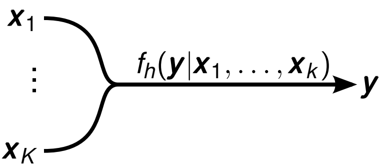
- $M \leq K + 1$: Number of time-shared rate vectors
- π_m: Permutation function of rate vector m
- $\alpha_m \geq 0$: Portion of time where rate vector m is used
- $\boldsymbol{Q}_k^{(m)}$: Covariance matrix of the k-th user signal in rate vector m
- Achievable rate of user k (including time-sharing):

$$r_k \leq \frac{1}{2} \sum_{m=1}^{M} \alpha_m \log \frac{\left| \boldsymbol{Z}_k + \boldsymbol{H}_k \left(\sum_{j \geq \pi_m^{-1}(k)} \boldsymbol{Q}_{\pi_m(j)}^{(m)} \right) \boldsymbol{H}_k^\dagger \right|}{\left| \boldsymbol{Z}_k + \boldsymbol{H}_k \left(\sum_{j > \pi_m^{-1}(k)} \boldsymbol{Q}_{\pi_m(j)}^{(m)} \right) \boldsymbol{H}_k^\dagger \right|}$$

- $\boldsymbol{Q}^{(m)} = \sum_{k=1}^{K} \boldsymbol{Q}_k^{(m)}$ must satisfy the power constraints for all m, and $\sum_m \alpha_m = 1$

Outline

Multiple access channel



- A multiple access channel is a probability transition function $f_h(\boldsymbol{y}|\boldsymbol{x}_1,\ldots,\boldsymbol{x}_K)$
- The capacity region was found in the early 70's [5, 6]
  - Let $\mathcal{S}\subseteq\{1,\ldots,K\}$, and $\bar{\mathcal{S}}=\{k|1\leq k\leq K, k\notin\mathcal{S}\}$
  - Let $\boldsymbol{x}(\mathcal{S})=\{\boldsymbol{x}_k|k\in\mathcal{S}\}$

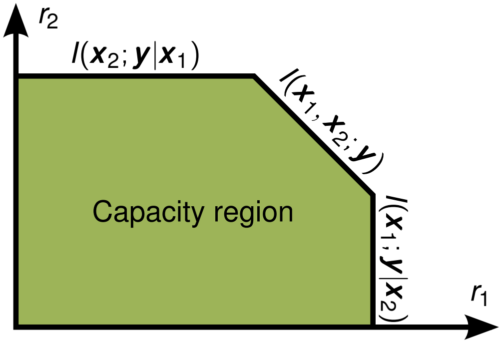
$$\sum_{k\in\mathcal{S}} r_k \leq I\left(\boldsymbol{x}\left(\mathcal{S}\right);\boldsymbol{y}|\boldsymbol{x}\left(\bar{\mathcal{S}}\right)\right), \quad \forall\mathcal{S}\subseteq\{1,\ldots,K\}$$

  - for certain $\prod_k f_{\boldsymbol{x}_k}(\boldsymbol{x}_k)$ such that all $f_{\boldsymbol{x}_k}$ satisfy the power constraints

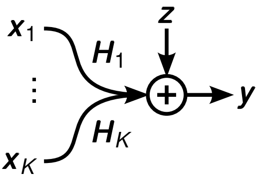
Multiple access channel

- For some $f_{\boldsymbol{x}_1}$ and $f_{\boldsymbol{x}_2}$, the two-user capacity region is

$$
\begin{aligned}
r_1 &\leq I(\boldsymbol{x}_1; \boldsymbol{y} | \boldsymbol{x}_2) \\
r_2 &\leq I(\boldsymbol{x}_2; \boldsymbol{y} | \boldsymbol{x}_1) \\
r_1 + r_2 &\leq I(\boldsymbol{x}_1, \boldsymbol{x}_2; \boldsymbol{y})
\end{aligned}
$$



Gaussian multiple access channel



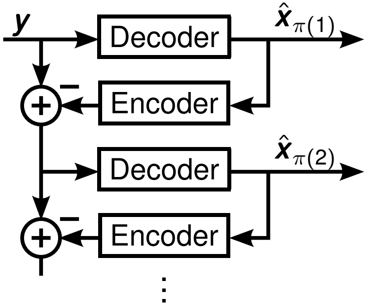
- Number of users: K
- Signal of user k: $\boldsymbol{x}_k$

$$\boldsymbol{y} = \sum_{k=1}^{K} \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{z}$$

- For all k, $\boldsymbol{Q}_k = \mathrm{E}[\boldsymbol{x}_k \boldsymbol{x}_k^\dagger]$ must satisfy certain power constraints
- *Successive interference cancelation* (SIC) and time-sharing achieve the capacity region
  - This is also true for the general multiple access channel (not only Gaussian)

Successive interference cancelation

- SIC detects the signal of each user sequentially
- The detected signal is subtracted before continuing with the next user



Successive interference cancelation

How does SIC work?

- The first user is interfered by user 2,3,4,...
- The second user is interfered by user 3,4,...
- The third user is interfered by user 4,...

- This scheme is equivalent to DPC
- The user ordering has to be optimized too
  - The optimum might be a combination of different orderings (time-sharing)

Successive interference cancelation

Achievable rates of one ordering

- K: Number of users
- $K!$: Number of user orderings
- $\boldsymbol{Q}_k$: Covariance matrix of the signal of user k
- $\boldsymbol{H}_k$: Channel matrix of user k
- $\boldsymbol{Z}$: Noise covariance matrix at the receiver
- π: users permutation function
- Achievable rate of user k in ordering π:

$$r_k \leq \frac{1}{2} \log \frac{\left|\boldsymbol{Z} + \sum_{j \geq \pi^{-1}(k)} \boldsymbol{H}_{\pi(j)} \boldsymbol{Q}_{\pi(j)} \boldsymbol{H}^{\dagger}_{\pi(j)}\right|}{\left|\boldsymbol{Z} + \sum_{j > \pi^{-1}(k)} \boldsymbol{H}_{\pi(j)} \boldsymbol{Q}_{\pi(j)} \boldsymbol{H}^{\dagger}_{\pi(j)}\right|}$$

Successive interference cancelation

Achievable rates of one ordering

- For the special case of the identity permutation function

$$r_1 \leq \frac{1}{2}\log\frac{\left|\boldsymbol{Z}+\sum_{j=1}^{K}\boldsymbol{H}_j\boldsymbol{Q}_j\boldsymbol{H}_j^\dagger\right|}{\left|\boldsymbol{Z}+\sum_{j=2}^{K}\boldsymbol{H}_j\boldsymbol{Q}_j\boldsymbol{H}_j^\dagger\right|}$$

$$r_2 \leq \frac{1}{2}\log\frac{\left|\boldsymbol{Z}+\sum_{j=2}^{K}\boldsymbol{H}_j\boldsymbol{Q}_j\boldsymbol{H}_j^\dagger\right|}{\left|\boldsymbol{Z}+\sum_{j=3}^{K}\boldsymbol{H}_j\boldsymbol{Q}_j\boldsymbol{H}_j^\dagger\right|}$$

$$\vdots$$

$$r_K \leq \frac{1}{2}\log\frac{\left|\boldsymbol{Z}+\boldsymbol{H}_K\boldsymbol{Q}_K\boldsymbol{H}_K^\dagger\right|}{\left|\boldsymbol{Z}\right|}$$

← Come back to DPC

Successive interference cancelation

Achievable rates of one ordering

$$r_1 + r_2 \leq \frac{1}{2} \log \frac{\left|\mathbf{Z} + \sum_{j=1}^{K} \mathbf{H}_j \mathbf{Q}_j \mathbf{H}_j^\dagger\right|}{\left|\mathbf{Z} + \sum_{j=2}^{K} \mathbf{H}_j \mathbf{Q}_j \mathbf{H}_j^\dagger\right|} + \frac{1}{2} \log \frac{\left|\mathbf{Z} + \sum_{j=2}^{K} \mathbf{H}_j \mathbf{Q}_j \mathbf{H}_j^\dagger\right|}{\left|\mathbf{Z} + \sum_{j=3}^{K} \mathbf{H}_j \mathbf{Q}_j \mathbf{H}_j^\dagger\right|}$$

$$= \frac{1}{2} \log \frac{\left|\mathbf{Z} + \sum_{j=1}^{K} \mathbf{H}_j \mathbf{Q}_j \mathbf{H}_j^\dagger\right|}{\left|\mathbf{Z} + \sum_{j=3}^{K} \mathbf{H}_j \mathbf{Q}_j \mathbf{H}_j^\dagger\right|}$$

$$\sum_{k=1}^{K} r_k \leq \frac{1}{2} \log \frac{\left|\mathbf{Z} + \sum_{j=1}^{K} \mathbf{H}_j \mathbf{Q}_j \mathbf{H}_j^\dagger\right|}{|\mathbf{Z}|}$$

- The sum rate is a concave function of the covariance matrices
  - If the power constraints are convex, the maximization of the sum rate is a convex problem

Successive interference cancelation

Achievable rates of SIC

- $M \leq K + 1$: Number of time-shared rate vectors
- π_m: Permutation function of rate vector m
- $\alpha_m \geq 0$: Portion of time where rate vector m is used
- $\boldsymbol{Q}_k^{(m)}$: Covariance matrix of the k-th user signal in rate vector m
- Achievable rate of user k (including time-sharing):

$$r_k \leq \frac{1}{2} \sum_{m=1}^{M} \alpha_m \log \frac{\left| \boldsymbol{Z} + \sum_{j \geq \pi_m^{-1}(k)} \boldsymbol{H}_{\pi_m(j)} \boldsymbol{Q}_{\pi_m(j)}^{(m)} \boldsymbol{H}_{\pi_m(j)}^{\dagger} \right|}{\left| \boldsymbol{Z} + \sum_{j > \pi_m^{-1}(k)} \boldsymbol{H}_{\pi_m(j)} \boldsymbol{Q}_{\pi_m(j)}^{(m)} \boldsymbol{H}_{\pi_m(j)}^{\dagger} \right|}$$

Outline

Duality

- Let f_h^B be the probability transition function of a GBC
- Let $\mathcal{P}^B$ be the set of feasible covariance matrices of a GBC
- Let $\mathcal{C}^B\left(f_h^B, \mathcal{P}^B\right)$ be the capacity region of the GBC
- Change the superscript B with M for the GMAC

- The GBC $\left(f_h^B, \mathcal{P}^B\right)$ and the GMAC $\left(f_h^M, \mathcal{P}^M\right)$ are dual if and only if

$$\mathcal{C}^B\left(f_h^B, \mathcal{P}^B\right) = \mathcal{C}^M\left(f_h^M, \mathcal{P}^M\right)$$

Duality

Why is duality important?

- If a rate vector is achievable in a GBC (GMAC), it is achievable in the dual GMAC (GBC)
- The optimum rate vector of a GBC (GMAC) is also optimum for the dual GMAC (GBC)

- It is usually easier to work with a GMAC than a GBC
  - We can look for rate vectors in the dual GMAC

Duality

Sum power constraint

GBC

- Sum power constraint:

$$\mathcal{P}^B = \left\{ (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_K) \middle| \operatorname{tr}\left(\sum_{k=1}^{K} \boldsymbol{Q}_k \right) \le P, \boldsymbol{Q}_k \succeq 0 \right\}$$

- All users with identity noise: $\boldsymbol{Z}_1 = \cdots = \boldsymbol{Z}_K = \boldsymbol{I}$

Dual GMAC

- Sum power constraint:

$$\mathcal{P}^M = \left\{ (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_K) \middle| \sum_{k=1}^{K} \operatorname{tr}(\boldsymbol{Q}_k) \le P, \boldsymbol{Q}_k \succeq 0 \right\}$$

- Noise $\boldsymbol{I}$, and hermitian channels $\boldsymbol{H}_k^M = \boldsymbol{H}_k^{B\dagger}$

Duality
Sum power constraint

- The optimum transmission parameters (orders, time-sharing weights and covariance matrices) can be computed from the dual GMAC
  - The optimum orders are reversed
  - The optimum time-sharing weights are the same
  - The optimum covariance matrices can be computed as in [7]

Duality

Linear power constraint

- Previous duality was generalized by Yu [8] to any linear power constraint
- The idea is to change the linear constraint with the noise

- The ideas of [8] can be used to further generalize duality to many linear constraints
- However this requires more elaboration (see [8] for more details)

GBC

- Linear power constraint:

$$\mathcal{P}^B = \left\{ (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_K) \middle| \operatorname{tr}\left(\boldsymbol{A} \sum_{k=1}^{K} \boldsymbol{Q}_k \right) \leq P, \boldsymbol{Q}_k \succeq 0 \right\}$$

- Users with any noise: $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_K$

Dual GMAC

- Linear power constraint:

$$\mathcal{P}^M = \left\{ (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_K) \middle| \sum_{k=1}^{K} \operatorname{tr}\left(\boldsymbol{Z}_k \boldsymbol{Q}_k \right) \leq P, \boldsymbol{Q}_k \succeq 0 \right\}$$

- Noise $\boldsymbol{A}$, and hermitian channels $\boldsymbol{H}_k^M = \boldsymbol{H}_k^{B\dagger}$

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[8] W. Yu, "Uplink-downlink duality via minimax duality," *IEEE Trans. Inf. Theory*, vol. 52, pp. 361–374, Feb. 2006.

Thank you for your attention